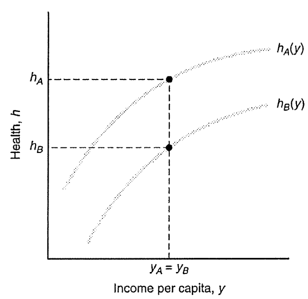
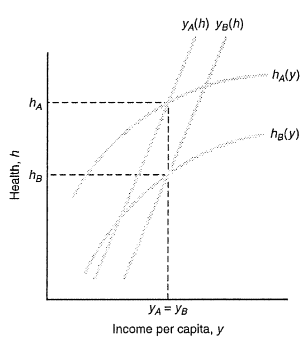
Eco 362: Economic Growth
Fall 2013
Solutions for Problem Set 3

Note: all Weil question numbers refer to the 3rd edition textbook.

Question 1: Chapter 6, Q2

Solution: Because for any given level of income Country A will generally be healthier than Country B, we can determine that the curve for Country A will be positioned above the curve for Country B. In addition, we observe that the income levels of both countries are identical. (The graphical depiction of this information is given by the first figure.) This implies that Country A achieves the same level of income as Country B, even though its workers are healthier. Consequently, it must be the case that the impact of health on income in Country A must be less, for any given level of health, than the impact on Country B. That is, will be positioned to the right of The equilibrium configuration is depicted in the second figure.





Question 2: In Weil Chapter 6, page 167, the author makes the following claim: "For developing countries, this calculation yields human capital's share of national income as 40% and for advanced countries it yields 45%". Using the data Table 6.2 and the returns to education (page 162 in the textbook), show how the author arrived at these numbers.

Solution:

The first step is to break up the wage for each level of education into the part that is payment for unskilled labor and the part that is payment for the skill due to education.

Highest level of Education	Years of schooling	Wages		Unskilled	Skilled
No schooling	0	1	1.00	1.00	0.00
Some Primary	4	1.134^4	1.65	1.00	0.65
Primary	8	1.134^4*1.101^4	2.43	1.00	1.43
Some Secondary	10	1.134^4*1.101^4^1.068^2	2.77	1.00	1.77
Secondary	12	1.134^4*1.101^4^1.068^4	3.16	1.00	2.16
Some higher	14	1.134^4*1.101^4^1.068^6	3.61	1.00	2.61
Higher	16	1.134^4*1.101^4^1.068^8	4.11	1.00	3.11

The second step uses the percentages of the population across the various education levels to calculate the fraction of total wage income that is payment for skills. We also know that the share of total income in the economy going to wages is $\approx \frac{2}{3}$ of national income. From this we can calculate the fraction of national income going towards skilled labor i.e. human capital's share of national income.

For developed countries:

Highest level of Education	Years of schooling	Wages	Unskilled	Skilled	% of pop Developed
No schooling	0	1.00	1.00	0.00	2.5
Some Primary	4	1.65	1.00	0.65	3.4
Primary	8	2.43	1.00	1.43	12.3
Some Secondary	10	2.77	1.00	1.77	17.8
Secondary	12	3.16	1.00	2.16	37.4
Some higher	14	3.61	1.00	2.61	9.9
Higher	16	4.11	1.00	3.11	16.6

Fraction of total wage income that is payment for skills$= \frac{209.66}{309.56} = 0.677$
Human capital Share$=\left(\frac{2}{3}\right)*$Skilled fraction$= 0.45$ of national income

For developing countries:

Highest level of Education	Years of schooling		Unskilled	Skilled	% of pop Developing
No schooling	0	1.00	1.00	0.00	20.8
Some Primary	4	1.65	1.00	0.65	10.4
Primary	8	2.43	1.00	1.43	18
Some Secondary	10	2.77	1.00	1.77	19.3
Secondary	12	3.16	1.00	2.16	23.2
Some higher	14	3.61	1.00	2.61	2.9
Higher	16	4.11	1.00	3.11	5.3

Fraction of total wage income that is payment for skills= $\frac{140.93}{240.836} = 0.415$
Human capital Share=$\left(\frac{2}{3}\right)$ *Skilled fraction= 0.39 of national income

Question 3: State the assumption of the Mankiw, Romer and Weil model of incorporating human capital into the basic Solow model. Derive the important equations. Note: the purpose of this question is so that you have the equations. You still need to be able to explain the predictions and work with the model as we discussed in class.

Solution:

Assumptions:

- Production function:

$$\begin{aligned} Y_t &= AK_t^{\alpha}\left(h_t^{\eta}L_t^{1-\eta}\right)^{1-\alpha} \\ h_t &: \text{ labor input per worker due to education} \\ L_t &: \text{ unskilled labor (i.e. population)} \end{aligned}$$

  where η is the human capital wage share of total *wages.*

- $L_{t+1} = (1+n)\,L_t$ for n constant
- Physical Capital: K_t : investment amounts to a fraction of output $s_K Y_t$ i.e. $I_t^K = s_K Y_t$ where s_K is constant
- Human capital (H_t) accumulation: here we make the following assumptions
  - Human capital depreciates at the same rate as physical capital (δ)
  - Investments in human capital need resources in the form of physical output (think of investment in schools etc.). And in particular, people invest a fixed fraction of their income ($s_H Y_t$) into human capital i.e. $I_t^H = s_H Y_t$ where s_H is constant

Consider the production function

$$\begin{aligned} Y_t &= AK_t^{\alpha}\left(h_t^{\eta}L_t^{1-\eta}\right)^{1-\alpha} \\ &\Rightarrow\; Y_t = AK_t^{\alpha}h_t^{\eta(1-\alpha)}L_t^{(1-\eta)1-\alpha} \\ &\Rightarrow\; Y_t = AK_t^{\alpha}h_t^{\beta}L_t^{1-\alpha-\beta} \end{aligned}$$

where $\beta = \eta(1-\alpha)$ and it is the share of human capital in total income (see earlier question)

Our production function has three inputs now and to make things simpler we make another assumption. The assumption is that K and h grow at the same rate. Suppose at time 0, when the economy starts out, the amounts of human capital and physical capital in the economy are fixed fractions of the income at time 0.

$$K_0 = s_K Y_0; h_0 = s_H Y_0$$

With the same depreciation rates and the shares of income allotted to physical and human capital remaining the same, this gives us that K and H grow at the same rate. (Recall the example with the red and blue shirts) This means

$$\frac{K_t}{h_t} = \frac{s_K}{s_H} = \frac{K_{t+1}}{h_{t+1}} = \frac{K_0}{H_0}$$

What this buys us is that we don't need to have data on h as we can back it out at every time from the data on K. i.e.

$$h_t = \frac{s_H}{s_K}K_t$$

Substituting this back into the production function gives

$$\begin{aligned} Y_t &= K_t^{\alpha}h_t^{\beta}L_t^{1-\alpha-\beta} \\ &= K_t^{\alpha}\left(\frac{s_H}{s_K}K_t\right)^{\beta}L_t^{1-\alpha-\beta} \\ &= \left(\frac{s_H}{s_K}\right)^{\beta}K_t^{\alpha+\beta}L_t^{1-\alpha-\beta} \end{aligned}$$

Suppose we denote

$$\left(\frac{s_H}{s_K}\right)^{\beta} = A$$

the production function becomes

$$Y_t = AK_t^{\alpha+\beta}L_t^{1-\alpha-\beta}$$

We now get something that looks very similar to the Solow model we considered earlier. The difference is that the coefficient on capital is $\alpha+\beta$. Remember, the capital share is still α, but since h_t is accumulated in the same way as K_t it is **as if** the share of capital is $\alpha+\beta$.

As always we start from the capital accumulation equation:

•

$$\begin{aligned}
K_{t+1} &= (1-\delta) K_t + s_K Y_t \\
&= (1-\delta) K_t + s_K \left(\frac{s_H}{s_K}\right)^{\beta} K_t^{\alpha+\beta} L_t^{1-\alpha-\beta} \\
&= (1-\delta) K_t + s_K^{1-\beta} s_H^{\beta} K_t^{\alpha+\beta} L_t^{1-\alpha-\beta} \\
\frac{K_{t+1}}{L_{t+1}} &= \frac{(1-\delta) K_t + s_K^{1-\beta} s_H^{\beta} K_t^{\alpha+\gamma} L_t^{1-\alpha-\beta}}{(1+n) L_t} \\
k_{t+1} &= \frac{(1-\delta) k_t + A s_K^{1-\beta} s_H^{\beta} k_t^{\alpha+\beta}}{(1+n)}
\end{aligned}$$

where $k = \frac{K_t}{L_t}$

Notice the importance of $\alpha + \beta$. If $\alpha + \beta < 1$ then we are in a world that looks similar to the Solow world with convergence. However it does reduce the speed of convergence, so in the short run we can talk about human capital and growth effects. See the classroom discussion for more details.

Assume in what follows that $\alpha + \beta < 1$. In this case our predictions for growth are qualitatively the same as the basic Solow model.

- Solving for the steady state

$$\begin{aligned}
k^{SS} &= \frac{(1-\delta) k^{SS} + s_K^{1-\beta} s_H^{\beta} \left(k^{SS}\right)^{\alpha+\beta}}{(1+n)} \\
k^{SS} &= \left(\frac{s_K^{1-\beta} s_H^{\beta}}{n+\delta}\right)^{\frac{1}{1-(\alpha+\beta)}}
\end{aligned}$$

•

$$y_t = \frac{Y_t}{L_t} = \frac{\left(\frac{s_H}{s_K}\right)^{\beta} K_t^{\alpha+\beta} L_t^{1-\alpha-\beta}}{L_t} = \left(\frac{s_H}{s_K}\right)^{\beta} k_t^{\alpha+\beta}$$

$$\begin{aligned}
y^{SS} &= \left(\frac{s_H}{s_K}\right)^{\beta} \left(k^{SS}\right)^{\alpha+\beta} \\
&= \left(\frac{s_H}{s_K}\right)^{\beta} \left(\frac{s_K^{1-\beta} s_H^{\beta}}{n+\delta}\right)^{\frac{\alpha+\beta}{1-(\alpha+\beta)}} \\
&= s_H^{\beta} s_K^{\alpha} \left(\frac{1}{n+\delta}\right)^{\frac{\alpha+\beta}{1-(\alpha+\beta)}}
\end{aligned}$$

Question 3: Consider the Solow model with exogenous technical progress where the production function is given by $Y_t = K_t^{\alpha} (e_t L_t)^{1-\alpha}$, $0 < \alpha < 1$,

a) Derive the equation for the evolution of $k_t = \frac{K_t}{e_t L_t}$

b) Derive the steady state of k

c) What is the growth rate of per capita income and total GDP?

d) Discuss the effects on the economy of a **one time, permanent** increase in the **growth rate of productivity** $(\hat{e})$ **rate**. Discuss the effects on y, Y graphically and in words.

Answer: Assumptions:

- The production function is given by $Y_t = K_t^{\alpha} (e_t L_t)^{1-\alpha}, 0 < \alpha < 1$
- $I_t = \gamma Y_t$ (for γ constant)
- $L_{t+1} = (1+n) L_t$ n assumed constant
- $e_{t+1} = (1+\hat{e}) e_t$ $\hat{e}$ assumed constant

Note: Make sure you know how to derive the equations we did in class because the derivations can be asked on the exams. Also make sure you know what the notation ($i.e. \delta, \gamma, n, \hat{e}, \alpha$) *stands for.*

a) We start with the capital accumulation equation

$$\begin{aligned}
K_{t+1} &= (1-\delta) K_t + I_t \\
K_{t+1} &= (1-\delta) K_t + \gamma Y_t \\
K_{t+1} &= (1-\delta) K_t + \gamma K_t^{\alpha} (e_t L_t)^{1-\alpha} \\
\frac{K_{t+1}}{e_{t+1} L_{t+1}} &= \frac{(1-\delta) K_t + \gamma K_t^{\alpha} (e_t L_t)^{1-\alpha}}{e_{t+1} L_{t+1}} \\
k_{t+1} &= \frac{(1-\delta) K_t + \gamma K_t^{\alpha} (e_t L_t)^{1-\alpha}}{(1+\hat{e}) e_t (1+n) L_t} \\
k_{t+1} &= \frac{1}{(1+\hat{e})(1+n)} \left[(1-\delta) \frac{K_t}{e_t L_t} + \gamma \frac{K_t^{\alpha} (e_t L_t)^{1-\alpha}}{e_t L_t} \right] \\
k_{t+1} &= \frac{(1-\delta) k_t + \gamma k_t^{\alpha}}{(1+\hat{e})(1+n)}
\end{aligned}$$

b) At the steady state $k_{t+1} = k_t = k^{SS}$. We also know that the relationship derived in part (a) holds at every k even at the steady state. Plugging in

$k_{t+1} = k_t = k^{SS}$ in to the equation derived above we get

$$
\begin{aligned}
k^{SS} &= \frac{(1-\delta) k^{SS} + \gamma \left(k^{SS}\right)^{\alpha}}{(1+\hat{e})(1+n)} \\
k^{SS}\left[(1+\hat{e})(1+n)\right] &= (1-\delta) k^{SS} + \gamma \left(k^{SS}\right)^{\alpha} \\
k^{SS}\left[1+\hat{e}+n+\hat{e}n-1+\delta\right] &= \gamma \left(k^{SS}\right)^{\alpha} \\
&\quad \text{for } \hat{e}, n \text{ small, } \hat{e}n \text{ is small enough to drop} \\
\left(k^{SS}\right)^{1-\alpha} &= \frac{\gamma}{\hat{e}+n+\delta} \\
k^{SS} &= \left(\frac{\gamma}{\hat{e}+n+\delta}\right)^{\frac{1}{1-\alpha}}
\end{aligned}
$$

c) Total GDP

$$
\begin{aligned}
Y_t &= K_t^{\alpha} \left(e_t L_t\right)^{1-\alpha} \\
&= \left(\frac{K_t}{e_t L_t}\right)^{\alpha} e_t L_t \\
Y_t &= k^{\alpha} e_t L_t
\end{aligned}
$$

Taking logs on both sides we get

$$\ln Y_t = \alpha \ln k + \ln e_t + \ln L_t$$

taking the derivative with respect to time on both sides we get

$$
\begin{aligned}
\frac{\frac{dY_t}{dt}}{Y_t} &= \alpha \frac{\frac{dk_t}{dt}}{k_t} + \frac{\frac{de_t}{dt}}{e_t} + \frac{\frac{dL_t}{dt}}{L_t} \\
\text{growth rate of } Y &= \alpha \left(\text{growth rate of } k\right) + \hat{e} + n
\end{aligned}
$$

In the Long run growth rate of $k = 0$ as it reaches the steady state so in the long run the growth rate of $Y = \hat{e} + n$. In the short run it may be above or below that depending on whether the growth rate of k is positive to negative during the transition to its steady state.

Per capita income

$$
\begin{aligned}
y_t &= \frac{Y_t}{L_t} = \frac{K_t^{\alpha} \left(e_t L_t\right)^{1-\alpha}}{L_t} \\
&= \frac{K_t^{\alpha} \left(e_t L_t\right)^{1-\alpha}}{L_t} \frac{e_t}{e_t} \\
&= \left(\frac{K_t}{e_t L_t}\right)^{\alpha} e_t \\
y_t &= k^{\alpha} e_t
\end{aligned}
$$

Taking logs on both sides we get

$$\ln y_t = \alpha \ln k + \ln e_t$$

taking the derivative with respect to time on both sides we get

$$\frac{\frac{dy_t}{dt}}{y_t} = \alpha\frac{\frac{dk_t}{dt}}{k_t} + \frac{\frac{de_t}{dt}}{e_t}$$
$$\text{growth rate of } Y = \alpha\,(\text{growth rate of } k) + \hat{e}$$

In the Long run, the growth rate of $k = 0$ as it reaches the steady state so in the long run the growth rate of $y = \hat{e}$ In the short run it may be above or below that depending on whether the growth rate of k is positive to negative during the transition to its steady state.

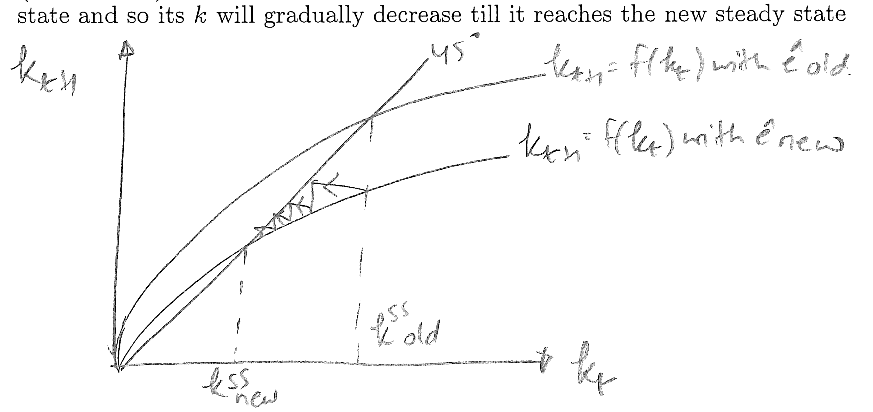
d) The economy is moving from a low productivity growth rate $(\hat{e}_{old})$ to a higher productivity growth rate $(\hat{e}_{new} > \hat{e}_{old})$.

Effects on $\tilde{k}$:

First let's start by discussing the effects on $\tilde{k}$ i.e. the capital to effective labor ratio, $\left(k_t = \frac{K_t}{e_t L_t}\right)$. The equation that governs the evolution of k is given by

$$k_{t+1} = \frac{(1-\delta)\,k_t + \gamma k_t^{\alpha}}{(1+n)\,(1+\hat{e})}$$

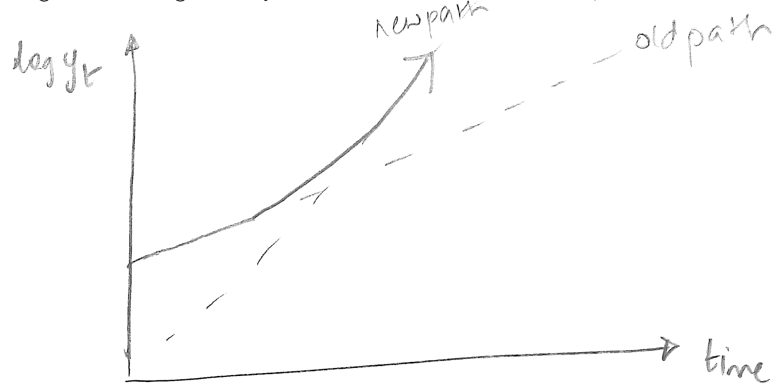
Consider a country starting out in steady state $\left(i.e. \text{ at } k_{old}^{SS}\right)$. An increase in $\hat{e}$ will shift the line that shows the relationship between k_{t+1} and k_t downwards. That is, for every k_t the economy will have a lower k_{t+1} with the higher productivity growth rate $(\hat{e}_{new})$ than it did with the old productivity growth rate $(\hat{e}_{old})$. The reason is that the population and investment behavior remains the same, but now the country has a number of effective workers in the future for every k_t. This implies that the steady state of the economy for k is now lower $\left(k_{new}^{SS} < k_{old}^{SS}\right)$. The economy is currently at higher k relative to its new steady state and so its k will gradually decrease till it reaches the new steady state



The long run effect on k is level effect of moving from a higher to a lower steady state level. The long run growth rate of k is still zero. In the short run it has negative growth till it reaches its new, higher steady state.

Effects on y :

Per-capita income (y) is a function of k and e_t $\left(y = e_t k_t^{\alpha}\right)$. It starts out growing along its old steady state growth path $\left(y^{SS}_{old}\right)$ and is growing at rate $\hat{e}$. When the productivity growth rate increases, k gradually decreases till it gets to its new lower steady state. However the $\hat{e}$ takes an immediate jump upwards to the new higher rate. Because of this, y grows slower than $\hat{e}_{new}$ till k gets to its new, lower steady state $\left(k^{SS}_{new}\right)$. After that y grows at the new higher rate of $\hat{e}_{new}$. The long run growth rate of y is now higher $\hat{e}_{new}$. In the short run its growth rate gradually increases till k reaches steady state.



Effects on Y :

While the country is in its old steady state, the long run growth rate of Y is given by $g = n + \hat{e}$. The easiest way to think through the effects of a change in $\hat{e}$ on Y is to remember the definition of per-capita income $\left(y = \frac{Y}{L}\right)$. We have been given that the growth rate of L remains unchanged So the denominator grows at the same rate compared to before the change. The growth rate of y is $\hat{e}_{new}$ in the long run. However the growth rate of y increases slowly to this new higher rate. In the short run, the numerator has to grow slower than its new higher steady state growth rate. From these observations we can deduce that the growth rate of Y will gradually increase from it old steady state growth rate of $n + \hat{e}_{old}$ along its transition as k heads to a lower steady state. Once k reaches its new steady state the growth rate Y reaches the new higher rate $n + \hat{e}_{new}$ Also, since we know that in the new steady state $\frac{K}{eL}$ is constant we know that the growth rate of the capital stock (K_t) is the same as the new higher growth rate of $e_t L_t$. Remember the intuition for why Y grows at the growth rate of (eL) comes from the constant returns to scale property of the production function.